16.485: Visual Navigation for Autonomous Vehicles (VNAV)	**Fall 2023**
Lecture 6: Quadrotor Dynamics	
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This lecture introduces the Newton-Euler equation, basic aerodynamic effects of rotating propellers and the dynamical model of a quadrotor. Furthermore we derive the derivative of a rotation matrix and discuss differential flatness of the quadrotor dynamics.

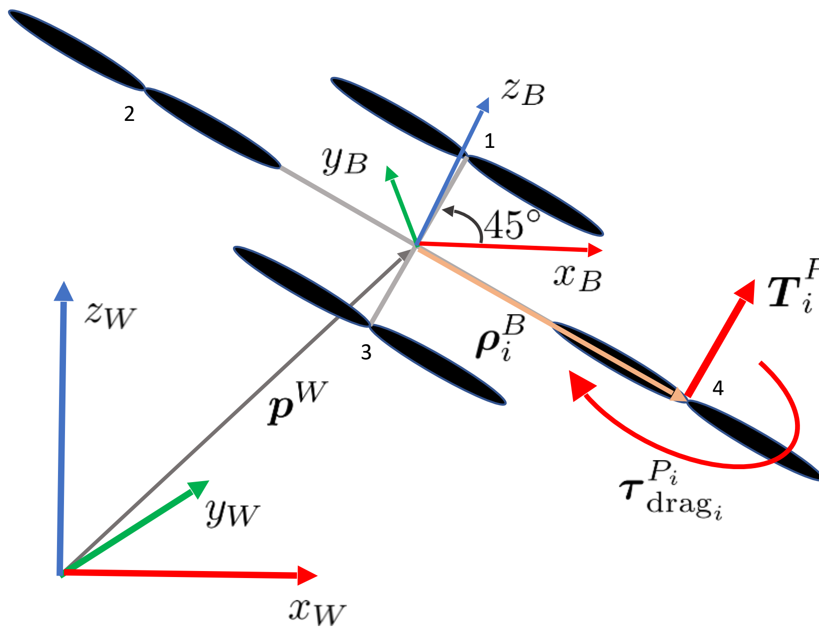

Figure 6.1: Quadrotor - most important variables are labeled.

6.1 Quadrotor model

Let us introduce the Newton-Euler equation, describing the translational and rotational dynamics of a rigid body:

$$\begin{bmatrix} \boldsymbol{f}^{\mathrm{W}} \\ \boldsymbol{\tau}^{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} m\boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\mathcal{J}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}^{\mathrm{W}} \\ \boldsymbol{\alpha}^{\mathrm{B}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\omega}^{\mathrm{B}} \times \boldsymbol{\mathcal{J}}\boldsymbol{\omega}^{\mathrm{B}} \end{bmatrix} \tag{6.1}$$

with symbols as defined in Table 6.1. Note that the translational part is the usual "$\boldsymbol{f} = m\boldsymbol{a}$" relation, while the rotational part is slightly more complicated. In particular, the extra cross product term at the end emerges when expressing the torque in the body frame (see `https://en.wikipedia.org/wiki/Euler%27s_equations_(rigid_body_dynamics)` for a justification).

$\boldsymbol{f}^{\mathrm{W}} \in \mathbb{R}^3$	applied total forces expressed in the world frame
$\boldsymbol{\tau}^{\mathrm{B}} \in \mathbb{R}^3$	applied total torques expressed in the body frame
$m \in \mathbb{R}_+$	body mass
$\boldsymbol{\mathcal{J}} \in \mathbb{R}^{3\times 3}$	moment of inertia about the center of mass
$\boldsymbol{I}_n$	$n \times n$ identity matrix
$\boldsymbol{a}^{\mathrm{W}} \in \mathbb{R}^3$	translational acceleration of the center of mass expressed in the world frame
$\boldsymbol{\alpha}^{\mathrm{B}} \in \mathbb{R}^3$	angular acceleration of the body expressed in the body frame
$\boldsymbol{\omega}^{\mathrm{B}} \in \mathbb{R}^3$	angular velocity of the body expressed in the body frame

Table 6.1: Symbol definitions

In the following, we discuss the nature of the forces $\boldsymbol{f}^{\mathrm{W}}$ and the torques $\boldsymbol{\tau}^{\mathrm{B}}$ when the rigid body is a quadrotor. The first external force we consider is *gravity*, which we make explicit in our model and move to the right-hand-side of (6.1) (note: $\boldsymbol{g}^{\mathrm{W}} \in \mathbb{R}^3$ is the gravity vector expressed in the world frame):

$$\begin{bmatrix} \boldsymbol{f}^{\mathrm{W}} \\ \boldsymbol{\tau}^{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} m\boldsymbol{I}_3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\mathcal{J}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}^{\mathrm{W}} \\ \boldsymbol{\alpha}^{\mathrm{B}} \end{bmatrix} + \begin{bmatrix} -m\boldsymbol{g}^{\mathrm{W}} \\ \boldsymbol{\omega}^{\mathrm{B}} \times \boldsymbol{\mathcal{J}}\boldsymbol{\omega}^{\mathrm{B}} \end{bmatrix} \tag{6.2}$$

Other forces and torques are due to aerodynamic effects. The two main aerodynamic effects for a quadrotor are:

- Thrust Force
- Rotor Drag

But there exist other effects

- Hub Force
- Hub Moment
- Rolling Moment
- Ground effect

At low speed ($<$ 10m/s), the latter are by more than one order of magnitude smaller than the rotor drag and the trust force, and can therefore be neglected. A more detailed explanation of these effects is given in [1].

Let us start modeling the *thrust force* of a full quadrotor. In a first approximation, the thrust force of a single rotor i, expressed in the reference frame of rotor i (a.k.a. propeller frame $_{P_i}$), can be computed as $\boldsymbol{T}_i^{P_i} = c_f w_i |w_i| \boldsymbol{e}_3$, with c_f being a constant coefficient, mapping the signed square of the rotor spinning velocity w_i to a force and $\boldsymbol{e}_3 = [0\ 0\ 1]^T$. This means that the thrust force pushes in the direction of the vertical axis of the propeller frame. For clarity, $w_i|w_i|$ is the signed square of the rotor spinning velocity and is sometimes denoted as $\text{sign}(w_i)(w_i)^2$. Therefore, the sum of all forces can be written as

$$\boldsymbol{f}^{B}_{thrust} = \sum_{i=1}^{4} \boldsymbol{R}^{B}_{P_i}\ \boldsymbol{T}_i^{P_i}, \tag{6.3}$$

with $\boldsymbol{R}^{B}_{P_i}$ being the rotation matrix from propeller- to body-frame ($\boldsymbol{R}^{B}_{P_i} = \mathbf{I}_3$ for standard quadrotors).

The second force is the *drag force* due to the airflow around the body of the quadrotor. This is typically modeled as a term proportional to the velocity of the quadrotor (we will include it in the model as part of the Lab 3 exercises), but for now we will assume that it is negligible at low speeds (we will instead consider the impact of drag on the torque).

Next we compute all torques. The torque consist of two components:

$$\boldsymbol{\tau}^{B} = \boldsymbol{\tau}^{B}_{drag} + \boldsymbol{\tau}^{B}_{thrust}, \tag{6.4}$$

the drag torque $\boldsymbol{\tau}^{B}_{drag}$ and thrust torque $\boldsymbol{\tau}^{B}_{thrust}$. The drag torque provided by every single propeller (expressed in the propeller frame $_{P_i}$) corresponds in a first approximation to $\boldsymbol{\tau}^{P_i}_{drag_i} = (-1)^{(i+1)} c_d w_i |w_i| \boldsymbol{e}_3$, with c_d being the propeller drag coefficient. The factor $(-1)^i$ is used since half of the propellers rotate clockwise and the other half rotates counter-clockwise. The total drag torque expressed in body frame is therefore

$$\boldsymbol{\tau}^{B}_{drag} = \sum_{i=1}^{4} \boldsymbol{R}^{B}_{P_i} \boldsymbol{\tau}^{P_i}_{drag_i} \tag{6.5}$$

The thrust torque results from applying a non-centered force (thrust force) to a body. From physics we know

$$\boldsymbol{\tau}^{B}_{thrust} = \sum_{i=1}^{4} (\boldsymbol{\rho}_i^{B} \times \boldsymbol{R}^{B}_{P_i} \boldsymbol{T}_i^{P_i}), \tag{6.6}$$

with $\boldsymbol{\rho}_i^{B}$ being the position (vector) of the propeller i in the body frame.

We can now include the forces and torques in the model (6.2) and rearrange the terms:

$$\begin{bmatrix} m\boldsymbol{a}^{W} \\ \boldsymbol{\mathcal{J}}\boldsymbol{\alpha}^{B} \end{bmatrix} = \begin{bmatrix} m\boldsymbol{g}^{W} \\ -\boldsymbol{\omega}^{B} \times \boldsymbol{\mathcal{J}}\boldsymbol{\omega}^{B} \end{bmatrix} + \begin{bmatrix} \boldsymbol{R}^{W}_{B} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{f}^{B}_{thrust} \\ \boldsymbol{\tau}^{B}_{drag} + \boldsymbol{\tau}^{B}_{thrust} \end{bmatrix} = \begin{bmatrix} -mg\boldsymbol{e}_3 \\ -\boldsymbol{\omega}^{B} \times \boldsymbol{\mathcal{J}}\boldsymbol{\omega}^{B} \end{bmatrix} + \begin{bmatrix} \boldsymbol{R}^{W}_{B} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_3 \end{bmatrix} \boldsymbol{F}\boldsymbol{w} \tag{6.7}$$

where we defined $\boldsymbol{w} = [w_1|w_1|,\ w_2|w_2|,\ w_3|w_3|,\ w_4|w_4|]^T$, we defined a suitable matrix $\boldsymbol{F}$, and we observed that for a typical choice of world frame (with the z axis pointing upwards) it holds $\boldsymbol{g}^{W} = -g\boldsymbol{e}_3$. We remark that the vector $\boldsymbol{F}\boldsymbol{w}$ essentially includes the forces (except gravity) and the torques in the body frame, which using (6.3)-(6.6) and assuming $\boldsymbol{R}^{B}_{P_i} = \mathbf{I}_3$, can be written as:

$$\begin{bmatrix} f^{B}_x \\ f^{B}_y \\ f^{B}_z \\ \tau^{B}_x \\ \tau^{B}_y \\ \tau^{B}_z \end{bmatrix} = \boldsymbol{F}\boldsymbol{w} = \begin{bmatrix} c_f\boldsymbol{e}_3 & c_f\boldsymbol{e}_3 & c_f\boldsymbol{e}_3 & c_f\boldsymbol{e}_3 \\ c_d\boldsymbol{e}_3 + c_f\boldsymbol{\rho}^{B}_1 \times \boldsymbol{e}_3 & -c_d\boldsymbol{e}_3 + c_f\boldsymbol{\rho}^{B}_2 \times \boldsymbol{e}_3 & c_d\boldsymbol{e}_3 + c_f\boldsymbol{\rho}^{B}_3 \times \boldsymbol{e}_3 & -c_d\boldsymbol{e}_3 + c_f\boldsymbol{\rho}^{B}_4 \times \boldsymbol{e}_3 \end{bmatrix} \boldsymbol{w} \tag{6.8}$$

Finally, we observe that (linear and angular) velocities and accelerations are related by:

$$\begin{aligned} \dot{\boldsymbol{p}}^{\mathrm{W}} &= \boldsymbol{v}^{\mathrm{W}} & \dot{\boldsymbol{v}}^{\mathrm{W}} &= \boldsymbol{a}^{\mathrm{W}} \\ \dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}} &= \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}[\boldsymbol{\omega}^{\mathrm{B}}]^{\wedge} & \dot{\boldsymbol{\omega}}^{\mathrm{B}} &= \boldsymbol{\alpha}^{\mathrm{B}} \end{aligned} \tag{6.9}$$

where $\boldsymbol{p}^{\mathrm{W}}$, $\boldsymbol{v}^{\mathrm{W}}$, $\boldsymbol{a}^{\mathrm{W}}$ are the position, velocity, and acceleration of the body frame "B" with respect to the "w" frame. Similarly, as mentioned above, $\boldsymbol{\omega}^{\mathrm{B}}$ and $\boldsymbol{\alpha}^{\mathrm{B}}$ are the angular velocity and acceleration of the body in the body frame. The only expression in (6.9) which is nontrivial is $\dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}} = \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}[\boldsymbol{\omega}^{\mathrm{B}}]^{\wedge}$, which we prove in the next section.

We can now use (6.7) and (6.9) to finalize the dynamical model of a quadrotor in terms of first-order differential equations:

$$\begin{aligned} \begin{bmatrix} m\dot{\boldsymbol{v}}^{\mathrm{W}} \\ \boldsymbol{\mathcal{J}}\dot{\boldsymbol{\omega}}^{\mathrm{B}} \end{bmatrix} &= \begin{bmatrix} -mg\boldsymbol{e}_3 \\ -\boldsymbol{\omega}^{\mathrm{B}} \times \boldsymbol{\mathcal{J}}\boldsymbol{\omega}^{\mathrm{B}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_3 \end{bmatrix} \boldsymbol{F}\boldsymbol{w} \\ \dot{\boldsymbol{p}}^{\mathrm{W}} &= \boldsymbol{v}^{\mathrm{W}} \\ \dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}} &= \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}[\boldsymbol{\omega}^{\mathrm{B}}]^{\wedge} \end{aligned} \tag{6.10}$$

where the *state* of the quadrotor is $\boldsymbol{p}^{\mathrm{W}}, \boldsymbol{v}^{\mathrm{W}}, \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}, \boldsymbol{\omega}^{\mathrm{B}}$ and the *control actions or inputs* are the propeller velocities, included in the vector $\boldsymbol{w}$.

6.1.1 Derivative of the rotation matrix

Consider the time-varying rotation matrix $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}} = \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)$, representing the rotation of the frame "B" with respect to frame "w". In the view of the orthogonality of $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}$, it holds:

$$\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)(\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t))^{\mathsf{T}} = \boldsymbol{I}_3 \tag{6.11}$$

which, differentiated with respect to time, gives

$$\dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t)(\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t))^{\mathsf{T}} + \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)(\dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t))^{\mathsf{T}} = \mathbf{0}. \tag{6.12}$$

Let us define the 3×3 matrix

$$\boldsymbol{S}(t) = \dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t)(\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t))^{\mathsf{T}}. \tag{6.13}$$

It is easy to show that $\boldsymbol{S}(t)$ is skew-symmetric since from (6.12) it follows:

$$\boldsymbol{S}(t) + \boldsymbol{S}(t)^{\mathsf{T}} = \mathbf{0}. \tag{6.14}$$

Postmultiplying both sides of (6.13) by $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)$ gives

$$\dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t) = \boldsymbol{S}(t)\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t) \tag{6.15}$$

Equation (6.15) relates the rotation matrix $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}$ to its derivative by means of the skew-symmetric operator $\boldsymbol{S}$. In order to get a physical insight on the nature of the skew-symmetric operator $\boldsymbol{S}$, let us consider a position $\boldsymbol{p}^{\mathrm{B}}$ and assume that this position is constant in the body frame. Clearly, we can express the same point in the world frame via: $\boldsymbol{p}^{\mathrm{W}}(t) = \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)\boldsymbol{p}^{\mathrm{B}}$. Now, let us compute the time derivative of $\boldsymbol{p}^{\mathrm{W}}(t)$ as (recall the point is fixed in the body frame):

$$\dot{\boldsymbol{p}}^{\mathrm{W}}(t) = \dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t)\boldsymbol{p}^{\mathrm{B}}, \tag{6.16}$$

which, in view of (6.13), can be expressed as

$$\dot{\boldsymbol{p}}^{\mathrm{W}}(t) = \boldsymbol{S}(t)\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)\boldsymbol{p}^{\mathrm{B}}. \tag{6.17}$$

If the vector $\boldsymbol{\omega}^{\mathrm{W}}(t)$ denotes the angular velocity of the body frame with respect to the world frame (expressed in the world frame), we know from basic physics that:

$$\dot{\boldsymbol{p}}^{\mathrm{W}}(t) = \boldsymbol{\omega}^{\mathrm{W}}(t) \times \boldsymbol{p}^{\mathrm{W}}(t) = \boldsymbol{\omega}^{\mathrm{W}}(t) \times \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)\boldsymbol{p}^{\mathrm{B}} \tag{6.18}$$

Therefore, comparing (6.17) and (6.18), it follows:

$$\boldsymbol{S}(t) = \boldsymbol{\omega}^{\mathrm{W}}(t)^{\wedge} \qquad \text{and} \qquad \dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t) = \boldsymbol{\omega}^{\mathrm{W}}(t)^{\wedge}\ \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t) \tag{6.19}$$

where the operator $(\cdot)^{\wedge}$, seen in previous lectures, transforms a vector to a skew symmetric matrix (recall that for two vectors $\boldsymbol{a}, \boldsymbol{b}$, it holds: $\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_{\times}\boldsymbol{b} = (\boldsymbol{a}^{\wedge})\boldsymbol{b}$.

Now in order to get the expression of the derivative of the rotation matrix in (6.9) we observe that for a vector $\boldsymbol{a}$ and a rotation matrix $\boldsymbol{R}$, it holds $(\boldsymbol{a})^{\wedge}\boldsymbol{R} = \boldsymbol{R}\ (\boldsymbol{R}^{\mathsf{T}}\boldsymbol{a})^{\wedge}$, from which it follows:[1]

$$\dot{\boldsymbol{R}}_{\mathrm{B}}^{\mathrm{W}}(t) = \boldsymbol{\omega}^{\mathrm{W}}(t)^{\wedge}\ \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t) = \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)(\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)^{\mathsf{T}}\boldsymbol{\omega}^{\mathrm{W}}(t))^{\wedge} = \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}(t)(\boldsymbol{\omega}^{\mathrm{B}}(t))^{\wedge} \tag{6.20}$$

6.2 Differential flatness property

A differentially flat system is one in which the state x and control inputs u can be expressed as functions of a subset of the system's outputs, called the flat outputs and their time-derivatives. In other words,

$$y = h(x, u, \dot{u}, \ddot{u}, \dots) \tag{6.21}$$

is a flat output if there exists smooth functions g_x and g_u such that

$$x = g_x(y, \dot{y}, \ddot{y}, \dots) \tag{6.22}$$

and

$$u = g_u(y, \dot{y}, \ddot{y}, \dots) \tag{6.23}$$

In this section we show that the quadrotor dynamics with the four angular velocities of the propellers $\boldsymbol{w}$ as inputs is differentially flat [2]. In other words, the states and the inputs can be written as algebraic functions of four carefully selected flat outputs and their derivatives. This facilitates the automated generation of trajectories since any smooth trajectory (with reasonably bounded derivatives) in the space of flat outputs can be followed by the underactuated quadrotor. Our choice of flat outputs is given by

$$\boldsymbol{\sigma} = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4]^{\mathsf{T}} = [x,\ y,\ z,\ \psi]^{\mathsf{T}}, \tag{6.24}$$

where $\boldsymbol{p}^{\mathrm{W}} = [x,\ y,\ z]^{\mathsf{T}}$ are the coordinates of the center of mass of the quadrotor in the world coordinate system and ψ is the yaw angle. We will define a trajectory, $\boldsymbol{\sigma}(t)$, as a smooth curve in the space of flat outputs:

$$\boldsymbol{\sigma}(t) : [t_o, t_m] \to \mathbb{R}^3 \times SO(2). \tag{6.25}$$

We will now show that the state of the system and the control inputs can be written in terms of $\boldsymbol{\sigma}$ and its derivatives. The following proof can be also found in [2].

Position $\boldsymbol{p}^{\mathrm{W}}$ and velocity $\boldsymbol{v}^{\mathrm{W}}$. The position and velocity of the center of mass are simply the first three terms of $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\sigma}}$, respectively.

[1] *Proof within the proof*: to show $(\boldsymbol{a})^{\wedge}\boldsymbol{R} = \boldsymbol{R}\ (\boldsymbol{R}^{\mathsf{T}}\boldsymbol{a})^{\wedge}$, we can show that the i-th column of the matrix in the left- and right-hand side of the equation are identical. The i-th column of $(\boldsymbol{a})^{\wedge}\boldsymbol{R}$ is $(\boldsymbol{a})^{\wedge}\boldsymbol{R}\boldsymbol{e}_i$ ($\boldsymbol{e}_i$ is a vector which is zero everywhere and has the i-th entry equal to 1). On the other hand, the i-th column of matrix in the rhs side is $\boldsymbol{R}(\boldsymbol{R}^{\mathsf{T}}\boldsymbol{a})^{\wedge}\boldsymbol{e}_i = \boldsymbol{R}(\boldsymbol{R}^{\mathsf{T}}\boldsymbol{a} \times \boldsymbol{e}_i) = \boldsymbol{a} \times \boldsymbol{R}\boldsymbol{e}_i = \boldsymbol{a}^{\wedge}\boldsymbol{R}\boldsymbol{e}_i$, concluding the proof.

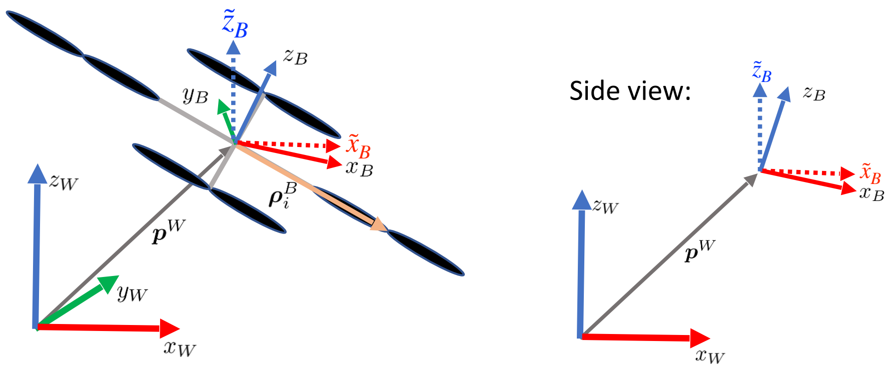

Figure 6.2: Quadrotor Frames to prove differential flatness.

Rotation matrix $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}$. To see that $\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}$ is a function of the flat outputs and their derivatives, consider the equation of motion (6.10). From (6.10), the translational part can be rewritten as:

$$m\boldsymbol{a}^{\mathrm{W}} = -mg\boldsymbol{e}_3 + f\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \iff \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} = \frac{m}{f}[\boldsymbol{a}^{\mathrm{W}} + g\boldsymbol{e}_3] \iff \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} = \frac{m}{f}\begin{bmatrix} \ddot{\sigma}_1 \\ \ddot{\sigma}_2 \\ \ddot{\sigma}_3 + g \end{bmatrix} \tag{6.26}$$

where we noticed that the thrust force (with some magnitude f) is applied along the vertical direction in the body frame, i.e., along the vector $\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}$. Noticing that $\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}$ must have unit norm, we realize we do not actually need to compute $\frac{m}{f}$, but we can simply write:

$$\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \qquad \text{with} \qquad \boldsymbol{v} = \begin{bmatrix} \ddot{\sigma}_1 \\ \ddot{\sigma}_2 \\ \ddot{\sigma}_3 + g \end{bmatrix} \tag{6.27}$$

Given the yaw angle, $\sigma_4 = \psi$, we can write the unit vector

$$\tilde{\boldsymbol{x}}_{\mathrm{B}}^{\mathrm{W}} = [\cos(\sigma_4), \sin(\sigma_4),\ 0]^{\mathsf{T}} \tag{6.28}$$

as shown in Figure 6.2. Note that $\tilde{\boldsymbol{x}}_{\mathrm{B}}^{\mathrm{W}}$ is an auxiliary vector describing the $\boldsymbol{x}$ axis of a rotating frame having the same yaw angle as B but with no pitch and roll. This auxiliary vector allows computing the $\boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}}$ axis as (provided $\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \times \tilde{\boldsymbol{x}}_{\mathrm{B}}^{\mathrm{W}} \neq \boldsymbol{0}$):

$$\boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}} = \frac{\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \times \tilde{\boldsymbol{x}}_{\mathrm{B}}^{\mathrm{W}}}{||\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \times \tilde{\boldsymbol{x}}_{\mathrm{B}}^{\mathrm{W}}||}, \tag{6.29}$$

from which we can also compute the last axis of the body frame:

$$\boldsymbol{x}_{\mathrm{B}}^{\mathrm{W}} = \boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}} \times \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}, \tag{6.30}$$

In other words, we uniquely determined

$$\boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}} = [\boldsymbol{x}_{\mathrm{B}}^{\mathrm{W}}\ \boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}}\ \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}] \tag{6.31}$$

provided we never encounter the singularity where $\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}$ is parallel to $\tilde{\boldsymbol{x}}_{\mathrm{B}}^{\mathrm{W}}$.

Angular velocity $\boldsymbol{\omega}^{\mathrm{B}}$. To show the angular velocity $\boldsymbol{\omega}^{\mathrm{B}}$ is a function of the flat outputs and their derivatives, take the first derivative of the first equation in (6.10):

$$\frac{d}{dt}\left(m\boldsymbol{a}^{\mathrm{W}}\right) = \frac{d}{dt}\left(-mg\boldsymbol{e}_3 + u_1\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}\right) \tag{6.32}$$

$$m\dot{\boldsymbol{a}}^{\mathrm{W}} = \frac{d}{dt}\left(u_1\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}\right) \tag{6.33}$$

$$m\dot{\boldsymbol{a}}^{\mathrm{W}} = \dot{u}_1\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} + \boldsymbol{\omega}^{\mathrm{W}} \times u_1\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}, \tag{6.34}$$

for some time-varying scalar u_1 (note: we leveraged again the fact that the thrust force is aligned with $\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}$). Projecting this expression along $\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}$, we obtain $\dot{u}_1 = \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \cdot m\dot{\boldsymbol{a}}^{\mathrm{W}}$ (the cross product vanishes in the projection). Substituting $\dot{u}_1$ back into (6.34):

$$m\dot{\boldsymbol{a}}^{\mathrm{W}} = (\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \cdot m\dot{\boldsymbol{a}}^{\mathrm{W}})\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} + \boldsymbol{\omega}^{\mathrm{W}} \times u_1 \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \iff m\dot{\boldsymbol{a}}^{\mathrm{W}} - m(\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \cdot \dot{\boldsymbol{a}}^{\mathrm{W}})\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} = u_1 \boldsymbol{\omega}^{\mathrm{W}} \times \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \tag{6.35}$$

Let us now define the vector $\boldsymbol{h}_\omega$ as

$$\boldsymbol{h}_\omega = \boldsymbol{\omega}^{\mathrm{W}} \times \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} = \frac{m}{u_1}(\dot{\boldsymbol{a}}^{\mathrm{W}} - (\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \cdot \dot{\boldsymbol{a}}^{\mathrm{W}})\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}). \tag{6.36}$$

$\boldsymbol{h}_\omega$ is the projection of $\frac{m}{u_1}\dot{\boldsymbol{a}}^{\mathrm{W}}$ onto the $\boldsymbol{x}_{\mathrm{B}}^{\mathrm{W}} - \boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}}$ plane, hence it can be easily expressed as a function of the flat outputs. We are now only left to relate $\boldsymbol{h}_\omega$ to the angular velocity $\boldsymbol{\omega}^{\mathrm{B}}$, or, equivalently, to $\boldsymbol{\omega}^{\mathrm{W}} = \boldsymbol{R}_{\mathrm{B}}^{\mathrm{W}}\boldsymbol{\omega}^{\mathrm{B}}$. If we write the body-frame components of the angular velocity (in the world frame) as $\boldsymbol{\omega}^{\mathrm{W}} = p\boldsymbol{x}_{\mathrm{B}}^{\mathrm{W}} + q\boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}} + r\boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}}$, then the components can be found as in [2]:

$$p = -\boldsymbol{h}_\omega \cdot \boldsymbol{y}_{\mathrm{B}}^{\mathrm{W}}, \qquad q = \boldsymbol{h}_\omega \cdot \boldsymbol{x}_{\mathrm{B}}^{\mathrm{W}} \qquad r = \dot{\psi}\boldsymbol{e}_3 \cdot \boldsymbol{z}_{\mathrm{B}}^{\mathrm{W}} \tag{6.37}$$

where $\dot{\psi} = \dot{\sigma_4}$ is the first derivative of the yaw angle.

Control inputs $\boldsymbol{\omega}$. We can repeat the reasoning in the previous paragraphs to prove that also the linear and angular accelerations are smooth functions of the flat outputs. Then, we can use the Newton-Euler equations (6.10) to conclude that also the inputs can be written as functions of $\boldsymbol{\sigma}$ in (6.24).

References

[1] Ilan Kroo, Fritz Prinz, Michael Shantz, Peter Kunz, Gary Fay, Shelley Cheng, Tibor Fabian, and Chad Partridge. The mesicopter: A miniature rotorcraft concept phase ii interim report. *Stanford university*, 2000.

[2] Daniel Mellinger and Vijay Kumar. Minimum snap trajectory generation and control for quadrotors. In *Robotics and Automation (ICRA), 2011 IEEE International Conference on*, pages 2520–2525. IEEE, 2011.